

# A Vector Non-abelian Chern-Simons Duality

H. García-Compeán<sup>a\*</sup>, O. Obregón<sup>b†</sup> and C. Ramírez<sup>c‡</sup>

<sup>a</sup> *Departamento de Física*

*Centro de Investigación y de Estudios Avanzados del IPN*

*P.O. Box 14-740, 07000, México D.F., México*

<sup>b</sup> *Instituto de Física de la Universidad de Guanajuato*

*P.O. Box E-143, 37150, León Gto., México*

<sup>c</sup> *Facultad de Ciencias Físico Matemáticas*

*Universidad Autónoma de Puebla*

*P.O. Box 1364, 72000, Puebla, México*

(February 1, 2008)

## Abstract

Abelian Chern-Simons gauge theory is known to possess a ‘S-self-dual’ action where its coupling constant  $k$  is inverted *i.e.*  $k \leftrightarrow \frac{1}{k}$ . Here a vector non-abelian duality is found in the pure non-abelian Chern-Simons action at the classical level. The dimensional reduction of the dual Chern-Simons action to two-dimensions constitutes a dual Wess-Zumino-Witten action already given in the literature.

hep-th/0103066

---

\*E-mail address: `compean@fis.cinvestav.mx`

†E-mail address: `octavio@ifug3.ugto.mx`

‡E-mail address: `cramirez@fcfm.buap.mx`

## I. INTRODUCTION

Duality is a very important tool in the study of nonperturbative physics in quantum field and string theories (for a review, see for instance [1]). In this context, duality helps to describe the strong coupling limit of some supersymmetric field and string theories. Thus, it is important to determine if a theory does admit dual versions. In order to do that, the Roček-Verlinde procedure is very useful [2]. One general signature to know whether a system can be described through ‘dual’ variables is the presence of a *global* symmetry. This symmetry can be made local to construct a more general Lagrangian with additional variables (Lagrange multiplier fields) and a bigger symmetry. From this *parent* Lagrangian, the original Lagrangian and its associated dual Lagrangian can be obtained. This global symmetry can be *abelian* or *non-abelian* and, according to it, the above mentioned dualization procedure is called, *abelian* or *non-abelian duality*. Abelian duality is nowadays well understood (for a review see for instance [3,4]). However non-abelian duality has a more complicated structure. Non-abelian duality was originally proposed by de la Ossa and Quevedo in [5]. Its global structure was investigated in [6] and further worked out in [7,8]. In particular in Ref. [8], the structure of non-abelian duality of Wess-Zumino-Witten (WZW) models was studied in detail at the quantum level. A non-trivial generalization of the non-abelian  $T$ -duality is the Poisson-Lie  $T$ -duality, which was considered by Klimčík and Severa in a series of papers [9].

On the other hand, Chern-Simons gauge theory has been used to describe a wide range of phenomena in three dimensions. This range from condensed matter systems in low dimensions and particularly in the fractional quantum Hall effect and superconductivity (see for instance [10]), to  $(2+1)$ -dimensional gravity [11]. On the mathematical side, Chern-Simons theory has been very useful for constructing knots and links invariants [12]. The study of duality in the *abelian* Chern-Simons action was firstly introduced in Ref. [13] and further studied in Refs. [14,15]. In particular, in Ref. [15], the effects of  $T$ -duality in the fractional quantum Hall effect were computed. This duality works by interchanging the level  $k$  of the Chern-Simons theory to  $\frac{1}{k}$ . The generalization to the non-abelian Chern-Simons and

supersymmetric Chern-Simons cases has been worked out in Refs. [16,17]. In these papers, it was found that at the classical level, the non-abelian dual theories are also non-abelian Chern-Simons theories with inverted level, just as in the abelian case. Recently, some new non-abelian dualities in two-dimensional models were discovered by reducing dimensionally certain three-dimensional non-abelian dual systems [18]. These results seem to be relevant to massive IIA supergravity. Another recent application of non-abelian duality concerns with the dual descriptions of Belavin-Polyakov instantons [19].

A very different duality of the Chern-Simons action has also been discussed by Kapustin and Strassler in the context of the mirror symmetry of the abelian gauge theory in three dimensions [20]. In this latter paper it is found that  $\mathcal{N} = 3$  Chern-Simons QED and  $\mathcal{N} = 4$  QED are in fact  $S$ -dual with the mapping  $k \leftrightarrow \frac{1}{k}$ . In this case the non-abelian vector duality generalization remain still as an open problem.

It is well known from Ref. [12] that for compact groups, the quantization of Chern-Simons gauge theory consists on the finite-dimensional Hilbert state constructed from the conformal blocks of the associated two-dimensional rational conformal field theory (RCFT). Other features of CFT, like the conformal anomaly and the duality of conformal blocks, can also be reinterpreted by means of the Chern-Simons three-dimensional theory [21]. It is natural to ask whether the Roček-Verlinde procedure for the WZW models can be carried over from the Chern-Simons theory perspective. In this paper we find a positive answer to this question.

We address the problem of non-abelian duality in the Chern-Simons gauge theory in three dimensions, with compact and simple gauge group. In the process we derive the non-abelian duality of WZW models found in Ref. [8], from the Chern-Simons perspective. To be specific, we exploit the fact that the Chern-Simons action is invariant under global transformations of the connection in the adjoint representation. This symmetry is gauged out, and the dual action is then obtained. In order to verify the proposed nonabelian Chern-Simons duality and its possible consequences, we reduce the parent action to its 2D counterpart. It turns out that it coincides with the duality found in Ref. [8], where consequences of it were computed.

The reduction to 2D RCFT may help us to understand properly the structure of nonabelian duality in three dimensions. In [22], mirror symmetry of 3D Chern-Simons theories with a geometric interpretation, realized as a brane configuration of D-branes and NS-branes, were found. Our results would be relevant to find a relation of [22] to the geometrical WZW models realized as brane configurations [23].

This paper is organized as follows: In section II we briefly review the necessary tools of non-abelian duality (we follow Ref. [4]) which will be useful in the subsequent sections. Sections III and IV are the main contribution of this paper. In section III basically we find the dual non-abelian Chern-Simons action. In Section IV we reduce the dual action to two dimensions. Finally in Section V we give our concluding remarks.

## II. NON-ABELIAN DUALITY

In this section we will briefly recall the basics of non-abelian duality. Non-abelian duality was first proposed in Ref. [5], in the context of the target space duality in string theory, and further developed in [6–9]. The starting point is a given non-linear sigma model described by a Lagrangian  $L$  depending of  $M$  world-sheet scalar fields  $X^M$  and with non-constant target space metric  $G_{MN}(X)$ . This metric is assumed to possess a group of non-abelian isometries  $\mathbf{G}$ . Let  $n$  be the index denoting the isometric directions. Then scalar fields transform under the global group  $\mathbf{G}$  as  $X^m \rightarrow g_n^m X^n$  with  $g_n^m \in \mathbf{G}$ . Following the Roček-Verlinde procedure, one can gauge out a non-abelian subgroup  $\mathbf{H}$  of  $\mathbf{G}$ , with  $\partial X^m \rightarrow \mathcal{D}X^m = \partial X^m + A^\alpha (T_\alpha)_n^m X^n$ . The procedure also incorporates to the action a term  $\int tr(\Lambda F)$  where  $F = \partial \bar{A} - \bar{\partial} A + [A, \bar{A}]$  and  $\Lambda$  is a two-indices Lagrange multiplier field. The gauge field is a Lie algebra  $(Lie(\mathbf{G}))$ -valued field in the adjoint representation of  $\mathbf{H}$ .

The partition function is given by

$$Z = \int \frac{\mathcal{D}X}{V_{\mathbf{G}}} \int \mathcal{D}\Lambda \mathcal{D}A \mathcal{D}\bar{A} \exp \left\{ -i \left( S_{gauged}[X, A, \bar{A}] + \int tr(\Lambda F) \right) \right\}. \quad (1)$$

The original action can, as usual, be found by integration of the Lagrange multiplier  $\Lambda$ . The dual theory can be obtained by integrating over the gauge fields  $A$  and  $\bar{A}$ . It yields

$$Z = \int \mathcal{D}X \mathcal{D}\Lambda \delta[\mathcal{F}] \det \frac{\delta \mathcal{F}}{\delta \omega} \exp \left( -iS'[X, \Lambda] \right) \det(f^{-1}), \quad (2)$$

where  $\mathcal{F}$  is the gauge fixing function,  $\omega$  represents the parameters of the group of isometries,  $f$  is a matrix-valued coefficient of the quadratic term in the gauge fields and  $S'[X, \Lambda]$  is given by

$$S'[X, \Lambda] = S[X] - \frac{1}{4\pi\alpha'} \int \bar{J}_\alpha (f^{-1})^{\alpha\beta} J_\beta, \quad (3)$$

where  $J$  and  $\bar{J}$  are currents coupled to  $\bar{A}$  and  $A$  respectively.

In the next section we will show that the non-abelian Chern-Simons theory possesses *exactly* this non-abelian duality structure. Thus, it will constitute a new example of this sort of duality.

### III. NON-ABELIAN CHERN-SIMONS DUAL ACTION

Consider the pure Chern-Simons action

$$L = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr} \left( AdA + \frac{2}{3} A^3 \right), \quad (4)$$

where  $A$  is a connection on the  $G$ -bundle  $E$  over  $\mathcal{M}$ ,  $G$  is a compact and simple Lie group and  $\mathcal{M}$  is an oriented arbitrary three-manifold with non-empty boundary  $\partial\mathcal{M} \neq \emptyset$ . ‘Tr’ is an invariant quadratic form on the Lie algebra  $\mathcal{G} = \text{Lie}(G)$  of  $G$ . The wedge product is omitted from this action.

Let  $\{T_a\}$ ,  $a = 1, \dots, \dim(G)$ , be a basis of  $\mathcal{G} = \text{Lie}(G)$  with  $[T^a, T^b] = f_c^{ab} T^c$  and ‘Tr’ the diagonal quadratic form  $\text{Tr}(T_a T_b) = -2\delta_{ab}$ . In local coordinates of  $\mathcal{M}$ , the action (4) is thus written as

$$L = \frac{k}{4\pi} \int_{\mathcal{M}} \varepsilon^{ijk} \left( A_i^a \partial_j A_{ka} + \frac{1}{3} f_{abc} A_i^a A_j^b A_k^c \right). \quad (5)$$

The partition function of this theory is given by

$$Z = \int \mathcal{D}A \exp\left(\frac{ik}{4\pi} \int_{\mathcal{M}} \text{Tr}(AdA + \frac{2}{3}A^3)\right). \quad (6)$$

We intend to find a “dual” action to (5) using the non-abelian generalization of the Roček-Verlinde procedure originally proposed in [5] and further developed in [6,8,24,25] (for some reviews see, [3,4].) We begin by noting that the action (5) is invariant under the global transformations

$$A_i \rightarrow g^{-1}A_i g, \quad (7)$$

where  $g$  is a constant element of  $G$ . As in the standard procedure, we gauge a subgroup  $H$  of the above symmetry in the action (5), with algebra  $\mathcal{H}$ , and introduce the  $\mathcal{H}$ -valued gauge field  $B_i$  to get the parent action:

$$L_D = \int_{\mathcal{M}} \varepsilon^{ijk} \left[ \frac{k}{4\pi} (A_i^a \widehat{D}_j A_{ka} + \frac{1}{3} f_{abc} A_i^a A_j^b A_k^c) + \frac{1}{2\pi} \chi_i^a F_{jka}(B) \right], \quad (8)$$

where the Lagrange multipliers  $\chi_i^a$  and the field strength  $F_{jk}^a(B) = \partial_j B_k^a - \partial_k B_j^a + f_{bc}^a B_j^b B_k^c$  are  $\mathcal{H}$ -valued forms on  $\mathcal{M}$ , and  $\widehat{D}_i = \partial_i + [B_i, \cdot]$  are the corresponding covariant derivatives with respect to the  $B$ -fields. This action is invariant under the symmetry transformations

$$B_i \rightarrow h^{-1} B_i h + h^{-1} \partial_i h, \quad A_i \rightarrow h^{-1} A_i h, \quad \chi \rightarrow h \chi h^{-1}, \quad (9)$$

that is,

$$\widehat{D}_i A = h^{-1} \widehat{D}_i A h, \quad (10)$$

for any element  $h$  of  $H$ .

The partition function for this system is given by

$$Z = \int \mathcal{D}A \mathcal{D}\chi \mathcal{D}B \exp\left(i L_D\right). \quad (11)$$

Integrating with respect to the Lagrange multipliers  $\chi_i^a$ , we get the constraints

$$F_{jk}^a(B) = 0, \quad (12)$$

which lead to consider only flat gauge connections  $B_i$  of the form

$$B_i = h^{-1} \partial_i h. \quad (13)$$

Then locally the gauge fields are pure gauge, the gauge fixing  $B_i^a = 0$  can be chosen, and we recover the original action (5).

On the other hand, the “dual” action can be obtained by integrating over the  $\mathcal{H}$ -valued gauge fields  $B_i^a$  and then fixing the gauge. The relevant part of the action is

$$L_D = \int_{\mathcal{M}} \varepsilon^{ijk} \left( \dots + \frac{k}{8\pi} f_{abc} A_i^a A_j^b B_k^c + \frac{1}{2\pi} \chi_i^a \partial_j B_{ka} + \frac{1}{4\pi} f_{abc} \chi_i^a B_j^b B_k^c + \dots \right). \quad (14)$$

This is a Gaussian integral and the integration defines the “dual” action  $L_D^*$  with partition function

$$Z = \int \mathcal{D}A \mathcal{D}\chi \det M^{-1/2} \exp(i L_D^*), \quad (15)$$

with the H-invariant dual action,

$$\begin{aligned} L_D^*[A, \chi] = & L[A] + \frac{1}{4\pi} \int_{\mathcal{M}} J_i^a(\chi, A) M^{-1}{}_{ab}{}^{ij} J_j^b(\chi, A) \\ & + \frac{1}{4\pi} \int_{\partial\mathcal{M}} N_a^i(\chi) M^{-1}{}_{ij}{}^{ab} \left( N_b^j(\chi) - 2J_b^j(\chi, A) \right), \end{aligned} \quad (16)$$

where  $L[A]$  is the standard Chern-Simons action (5),

$$J_a^k(\chi, A) = \varepsilon^{ijk} \left( \partial_i \chi_{ja} + \frac{k}{8\pi} f_{abc} A_i^b A_j^c \right), \quad (17)$$

$N_a^i = \varepsilon^{ijk} n_j \chi_{ka}$ , with  $n_i$  the normal to  $\partial\mathcal{M}$ , and  $M^{-1}$  is the inverse matrix of

$$M_{ab}^{ij} = \frac{1}{2} \varepsilon^{ijk} f_{abc} \chi_k^c. \quad (18)$$

Up to boundary terms, the form of the dual action (16) coincides with (3), with the difference that the factor determinant in (15), appears here with a square root due to the fact that the integral is Gaussian. The action (16) is still gauge invariant under  $H$  transformations and as usual, in order to obtain the true dual action, a gauge fixing has to be undertaken. However, as pointed out in [5] the dual action shall not depend the definite form of the gauge fixing.

The matrix (18) can be in general singular. Algebraically its inverse can be given, although not in a general simple form. For example for  $H=\text{SU}(2)$  it is given by

$$M^{-1}_{ij}{}^{ab} = \frac{1}{\det(M)} (\chi_i^a \chi_j^b - 2\chi_j^a \chi_i^b).$$

In the case  $\det(M)$  has singularities, counterterms have to be added in order to regularize the corresponding poles [24].

#### IV. DUAL WESS-ZUMINO-WITTEN ACTION

In order to understand the precise structure of the three-dimensional dual action (16), we compute the corresponding theory on the boundary  $\partial\mathcal{M}$  of  $\mathcal{M}$ . To do this we follow Refs. [26,27]. If the manifold  $\mathcal{M}$  has a boundary, we can consider the two dimensional theory corresponding to the action (8). In order to do that, we separate the time from the space components:  $d = d_0 + \tilde{d}$ ,  $A = A_0 + \tilde{A}$ ,  $B = B_0 + \tilde{B}$  and  $\chi = \chi_0 + \tilde{\chi}$ , then the parent action (8) can be rewritten as

$$L_D = \int_{\mathcal{M}} \frac{1}{4\pi} \text{Tr} \left[ -\frac{1}{2} \tilde{A} d_0 \tilde{A} + \tilde{\chi} d_0 \tilde{A} + \chi_0 \tilde{F}(\tilde{B}) - k A_0 (\tilde{d} \tilde{A} + \tilde{B} \tilde{A} + \tilde{A} \tilde{B} + \tilde{A}^2) + B_0 (\tilde{d} \tilde{\chi} + \tilde{B} \tilde{\chi} + \tilde{\chi} \tilde{B} - k \tilde{A}^2) \right], \quad (19)$$

where we set the boundary conditions  $A_0 = B_0 = 0$ .

Thus, after integrating out the Lagrange multipliers, we have the following equations:

$$\tilde{F}(\tilde{B}) = \tilde{d} \tilde{B} + \tilde{B}^2 = 0, \quad (20)$$

$$G(\tilde{A}, \tilde{B}) \equiv \tilde{d} \tilde{A} + \tilde{B} \tilde{A} + \tilde{A} \tilde{B} + \tilde{A}^2 = 0, \quad (21)$$

$$H(\tilde{A}, \tilde{B}, \tilde{\chi}) \equiv \tilde{d} \tilde{\chi} + \tilde{B} \tilde{\chi} + \tilde{\chi} \tilde{B} - k \tilde{A}^2 = 0. \quad (22)$$

Further, we observe that if we define  $\tilde{A} = \tilde{\phi} - \tilde{B}$  and  $\tilde{\chi} = -k\tilde{A} + \tilde{\lambda}$ , where  $\tilde{\phi} \in \mathcal{G}$  and  $\tilde{\lambda} \in \mathcal{H}$ , then  $G(\tilde{A}, \tilde{B}) = \tilde{F}(\tilde{\phi}) - \tilde{F}(\tilde{B})$  and  $H(\tilde{A}, \tilde{B}, \tilde{\chi}) = -kG(\tilde{A}, \tilde{B}) + \tilde{d}\tilde{\lambda} + \tilde{B}\tilde{\lambda} + \tilde{\lambda}\tilde{B}$ .



Therefore the equations we have to set to zero are  $\tilde{F}(\tilde{B}) = F(\tilde{\phi}) = 0$  and  $\tilde{d}\tilde{\lambda} + \tilde{B}\tilde{\lambda} + \tilde{\lambda}\tilde{B} = 0$ . The two first equations can be solved by  $\tilde{B} = h^{-1}\tilde{d}h$  and  $\tilde{\phi} = g^{-1}\tilde{d}g$ , where  $h \in H$  and  $g \in G$ . Now, if we insert these solutions into the last equation, we get,

$$\tilde{d}\tilde{\lambda} + h^{-1}\tilde{d}h\tilde{\lambda} + \tilde{\lambda}h^{-1}\tilde{d}h = 0 \quad (23)$$

that is

$$h\tilde{d}\tilde{\lambda}h^{-1} + \tilde{d}h\tilde{\lambda}h^{-1} + h\tilde{\lambda}h^{-1}\tilde{d}hh^{-1} = \tilde{d}(h\tilde{\lambda}h^{-1}) = 0. \quad (24)$$

Therefore

$$\tilde{\lambda} = h^{-1}\tilde{d}\alpha h, \quad \tilde{\chi} = -k\tilde{A} + h^{-1}\tilde{d}\alpha h \quad (25)$$

Where  $\alpha \in \mathcal{H}$ . We get,

$$L_D = \int_{\mathcal{M}} \frac{1}{4\pi} \text{Tr} \left[ \frac{k}{2} g^{-1}\tilde{d}gd_0(g^{-1}\tilde{d}g) - \frac{k}{2} h^{-1}\tilde{d}hd_0(h^{-1}\tilde{d}h) + h^{-1}\tilde{d}\alpha hd_0(h^{-1}\tilde{d}h) \right], \quad (26)$$

which can be rewritten as

$$L_D = \int_{\mathcal{M}} \frac{1}{4\pi} \text{Tr} \left[ \tilde{d} \left( \frac{k}{2} \tilde{d}g^{-1}d_0g - \frac{k}{2} \tilde{d}h^{-1}d_0h + \tilde{d}\alpha d_0hh^{-1} \right) - \frac{k}{6} (h^{-1}dh)^3 + \frac{k}{6} (g^{-1}dg)^3 \right]. \quad (27)$$

Therefore, if for example our manifold is  $\mathcal{M} = \mathbf{R} \times \mathbf{D}$ , where  $\mathbf{D}$  is a 2-disk, then if  $r$  and  $\phi$  are the coordinates on the disk, we get the two dimensional parent action,

$$\begin{aligned} I_D = \int_{\mathbf{R} \times \partial \mathbf{D}} \frac{1}{4\pi} \text{Tr} \left( -\frac{k}{2} g^{-1} \partial_\phi g g^{-1} \partial_t g + \frac{k}{2} h^{-1} \partial_\phi h h^{-1} \partial_t h - \partial_\phi \alpha \partial_t h h^{-1} \right) d\phi dt \\ + \frac{k}{24\pi} \int_{\mathbf{R} \times \mathbf{D}} \text{Tr} \left[ (g^{-1}dg)^3 - (h^{-1}dh)^3 \right]. \end{aligned} \quad (28)$$

This parent action contains two WZW actions for  $g$  and  $h$ , as well as the  $\alpha$ -term. It coincides with the nonabelian duality parent action for WZW given in [8] (see Eq. (4.16) of Ref. [8]).

From (25), we see that the field  $\alpha$  corresponds to the field  $\chi$  in (8) and thus its integration should give the WZW action as result. Indeed, the integration over  $\alpha$  gives  $\partial_\phi(h^{-1}\partial_t h) = 0$ , whose solution is  $h(t, \phi) = A(\phi)B(t)$ . After substitution of this solution back into the parent

action (28), the  $\alpha$ -term and the  $h$  WZW action vanish identically. Thus, as expected, the resulting action is the WZW action corresponding to the Chern-Simons action (4),

$$I_{WZW} = \frac{k}{2} \int \text{Tr} \, g^{-1} \partial_\phi g g^{-1} \partial_t g d\phi dt + \frac{k}{6} \text{Tr} \int (g^{-1} dg)^3. \quad (29)$$

In order to get the dual action to (29), it has to be integrated over the  $h$  field in (28) [8]. Let us denote by  $I_h$  the action containing  $h$ , its variation with respect to  $h$  gives

$$\delta_h I_h = -\frac{1}{4\pi} \int \text{Tr} \left[ h^{-1} \delta h \left( \partial_t \partial_\phi \alpha - k \partial_\phi (h^{-1} \partial_t h) + [h^{-1} \partial_t h, \partial_\phi \alpha] \right) \right] dt d\phi = 0, \quad (30)$$

which is solved by

$$\partial_\phi \alpha = k h^{-1} \partial_\phi h, \quad (31)$$

whose substitution in the parent action gives,

$$\begin{aligned} \tilde{I}_{WZW} = & -\frac{1}{4\pi} \int_{\mathbf{R} \times \partial \mathbf{D}} \text{Tr} \left( \frac{k}{2} g^{-1} \partial_\phi g g^{-1} \partial_t g + \frac{k}{2} h^{-1} \partial_\phi h h^{-1} \partial_t h \right) d\phi dt \\ & + \frac{k}{24\pi} \int_{\mathbf{R} \times \mathbf{D}} \text{Tr} \left[ (g^{-1} dg)^3 - (h^{-1} dh)^3 \right]. \end{aligned} \quad (32)$$

A detailed analysis done in [8], taking into account quantum corrections, in particular the ones due to the change of variable (31), shows that this dual nonabelian WZW action corresponds exactly to a  $(G/H)_k \times H_k$  WZW model.

## V. CONCLUDING REMARKS

In this paper we have further investigated the structure of non-abelian duality. We have found that one can associate a dual action to the non-abelian Chern-Simons action, which constitutes a new example of this kind of duality. After solving the constraints, we find an explicit ‘dual’ action for the original Chern-Simons action (5) in terms of a the Lagrange multiplier variables  $\chi_k^a$ , given by Eq. (16). This action contains a Chern-Simons term of the original fields  $A_t^a$  plus the  $\chi$  action and a boundary term. It needs still to be gauge fixed

in order to eliminate redundant degrees of freedom. In order to see which 2D CFT theory corresponds to this dual, we resort to the two dimensional theory on the boundary  $\mathbf{R} \times \partial\mathbf{D}$  corresponding to the parent action. This procedure was performed in the Sec. IV and the non-abelian action  $I_D$  obtained in Ref. [8] turned out.

We can summarize our results with the following diagram:

$$\begin{array}{ccc} \text{CS} & \xrightarrow{D} & \widetilde{\text{CS}} \\ \downarrow R & & \downarrow R \\ \text{WZW} & \xrightarrow{D} & \widetilde{\text{WZW}} \end{array}$$

where the mapping consisting in obtaining the dual action is denoted by  $D$ .  $R$  denotes the dimensional reduction of the dual action on the boundary  $\mathbf{R} \times \partial\mathbf{D}$ . The reduced parent action  $I_D$  Eq. (28) corresponds to two coupled WZW actions, just as was found in Ref. [8]. It is interesting to see that the above diagram commutes. The reason of this is that the dimensional reduction of  $\widetilde{\text{CS}}$  theory, *i.e.*  $\widetilde{\text{WZW}}$  model, coincides with the dual action  $\tilde{I}_{\text{WZW}}$  to the WZW model obtained in [8] which comes directly from dimensional reduction of the  $\text{CS}$  theory according to Refs. [26,27] and it is given by Eq. (32). This is so because both approaches have the same WZW parent action  $I_D$  Eq. (28).

It was the main aim of the paper to construct the non-abelian dual theory of the non-abelian Chern-Simons theory. Similarly to another examples, the dual action is a Chern-Simons action coupled to a Freedman-Townsend-like action Eq. (16). To find the utility of the non-abelian Chern-Simons duality, it remains to apply it to some systems involving non-abelian Chern-Simons theory. One of these examples would be to find a relation between some properties in the strong/weak coupling region of the Chern-Simons gauge theory to the weak/strong one of the same theory. This could be of importance, for instance, relate the topological invariants of knots and links defined in the strong coupling limit  $\frac{1}{k} \rightarrow \infty$  (Jones polynomial) and that defined in the weak coupling limit  $\frac{1}{k} \rightarrow 0$  (Vassiliev invariants).

Finally, non-abelian Chern-Simons action with non-compact complex groups are relevant in the description of  $(2+1)$  quantum gravity [11]. Dual actions for gravity and supergravity

were found in [16,17]. We would like to apply the issues considered here, to the Chern-Simons (super)gravity case and compare with the dual actions obtained in [16,17]. From Ref. [21] it is known that for the gravitational case, the associated Hilbert space is infinite dimensional. Even in this case CFT is of extreme importance to describe the gravity system [28]. Thus non-abelian Chern-Simons duality and its two-dimensional reduction would be useful to address the gravitational case.

Also, it is well known that non-abelian Chern-Simons gauge theory can be regarded as a topological string theory [29]. It is tantalizing to apply the dual Chern-Simons action in order to look for  $S$ -duality structure in the various involved topological sigma models. It would be also interesting to compare our results with that obtained by Mohammadi [30]. Some of these subjects are now under current investigation.

### Acknowledgments

This work was supported in part by CONACyT grants 28454E and 33951E.

## REFERENCES

- [1] A. Giveon and D. Kutasov, Rev. Mod. Phys. **71** (1999) 983, hep-th/9802067; A. Sen, “An Introduction to Non-perturbative String Theory”, hep-th/9802051.
- [2] M. Roček and E. Verlinde, “Duality, Quotients, and Currents”, Nucl.Phys. B **373** (1992) 630.
- [3] E. Alvarez, L. Alvarez-Gaumé, and Y. Lozano, “An Introduction to T-duality in String Theory”, Nucl. Phys. Proc. Suppl. 41 (1995) 1-20, hep-th/9410237.
- [4] F. Quevedo, “Duality and Global Symmetries”, Nucl. Phys. Proc. Suppl. **61** A (1998) 23.
- [5] X.C. de la Ossa and F. Quevedo. “Duality Symmetries from Non-abelian Isometries in String Theory”, Nucl. Phys. B **403** (1993) 377.
- [6] E. Alvarez, L. Alvarez-Gaumé, J.L.F. Barbón and Y. Lozano, “Some Global Aspects of Duality in String Theory”, Nucl. Phys. B **415** (1994) 71, hep-th/9309039.
- [7] A. Giveon and M. Roček, “On Non-abelian Duality”, Nucl.Phys. B **421** (1994) 173, hep-th/9308154.
- [8] E. Alvarez, L. Alvarez-Gaumé, and Y. Lozano, “On Non-abelian Duality”, Nucl. Phys. B **424** (1994) 155, hep-th/9403155.
- [9] C. Klimčik and P. Severa, Phys. Lett. B **351** (1995) 455; Phys. Lett. B **372** (1996) 65; C. Klimcik, Nucl. Phys. Proc. Suppl. **46** (1996) 116.
- [10] F. Wilczek (ed), *Fractional Statistics and Anyon Superconductivity*, World Scientific Singapore (1990).
- [11] E. Witten, “2 + 1 Dimensional Gravity as an Exactly Soluble System”, Nucl. Phys. B **311** (1988) 46.
- [12] E. Witten, Commun. Math. Phys. **121** (1989) 351.

- [13] A. Shapere and F. Wilczek, Nucl. Phys. B **320** (1989) 669; S.J. Rey and A. Zee, Nucl. Phys. B **352** (1991) 897.
- [14] D.-H. Lee, S. Kivelson, and S.-C. Zhang, Phys. Rev. Lett. **68** (1992) 2386; S. Kivelson, D.-H. Lee and S.-C. Zhang, Phys. Rev. B **46** (1992) 2223; C.A. Lütken and G.G. Ross, Phys. Rev. B **45** (1992) 11837; Phys. Rev. B **48** (1993) 2500; C.P. Burgess and B.P. Dolan, “Particle-Vortex Duality and the Modular Group: Applications to the Quantum Hall Effect and Other 2D Systems”, hep-th/0010246.
- [15] A.P. Balachandran, L. Chandar and B. Sathiapalan, Int. J. Mod. Phys. A **11** (1996) 3587.
- [16] H. García-Compeán, O. Obregón, C. Ramírez and M. Sabido, “Remarks on  $2 + 1$  Self-dual Chern-Simons Gravity”, Phys. Rev. D, **61** (2000) 0850022-1.
- [17] H. García-Compeán, O. Obregón, C. Ramírez and M. Sabido, “On  $S$ -duality in  $(2 + 1)$ -Chern-Simons Supergravity ”, Phys. Rev. D **64** (2001) 024002-1.
- [18] H. Gustafsson, S.E. Hjelmeland, U. Lindstrom, “Some New Non-Abelian 2D Dualities”, Phys. Scripta 60 (1999) 305, hep-th/9805072.
- [19] A. Pinzul and A. Stern, “Dual Instantons”, Phys. Rev. Lett. **85** (2000) 1374.
- [20] A. Kapustin and M.J. Strassler, “On Mirror Symmetry in Three Dimensional Abelian Gauge Theories”, hep-th/9902033.
- [21] E. Witten, “The Central Charge in Three Dimensions”, *Physics and Mathematics of Strings* eds. L. Brink, D. Friedan and A.M. Polyakov, World Scientific (1990).
- [22] K. Intriligator and N. Seiberg, “Mirror Symmetry in Three Dimensional Gauge Theories”, Phys. Lett. B **387** (1996) 513.
- [23] J. Maldacena, G. Moore and N. Seiberg, “Geometrical Interpretation of D-branes in Gauged WZW Models”, hep-th/0105038.

- [24] O. Ganor and J. Sonnenschein, *Int. J. Mod. Phys. A* **11** (1996) 5701.
- [25] N. Mohammadi, hep-th/9507040; Y. Lozano, *Phys. Lett. B* **364** (1995) 19.
- [26] G. Moore and N. Seiberg, “Taming the Conformal Zoo”, *Phys. Lett. B* **220** (1989) 422.
- [27] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, “Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory”, *Nucl. Phys. B* **326** (1989) 108.
- [28] M. Natsuume and Y. Satoh, “String Theory on Three Dimensional Black Holes, hep-th/9611041.
- [29] E. Witten, “Chern-Simons Gauge Theory as a String Theory”, hep-th/9207094.
- [30] N. Mohammadi, “On the classical connection between the WZWN model and topological gauge theories with boundaries”, *Phys. Rev. D* **62** (2000) 026005, hep-th/0001114.